A squeezing property and its applications to a description of long time behaviour in the 3D viscous primitive equations

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November 20, 2012

Abstract

We consider the 3D viscous primitive equations with periodic boundary conditions. These equations arise in the study of ocean dynamics and generate a dynamical system in a Sobolev H^1 type space. Our main result establishes the so-called squeezing property in the Ladyzhenskaya form for this system. As a consequence of this property we prove (i) the finiteness of the fractal dimension of the corresponding global attractor, (ii) the existence of finite number of determining modes, and (iii) ergodicity of a related random kick model. All these results provide a new information concerning long time dynamics of oceanic motions.

Keywords: 3D viscous primitive equations, periodic boundary conditions, global attractor, finite dimension, random kicks, ergodicity.
2010 MSC: 35Q35, 35B41, 76F25, 76F55

1 Introduction

We deal with the 3D viscous primitive equations which arise in geophysical fluid dynamics for modeling large scale phenomena in oceanic motions. These equations are based on the so-called hydrostatic approximation of the 3D Navier-Stokes equations for velocity field u which also contains a rotational (Coriolis) force and coupled to thermo- and salinity diffusion-transport equations (see, e.g., the survey [24] and the references therein). In this model a small variation of the density ρ of the fluid is taken into account via the buoyancy term only and has the form $\rho = \rho_0 - \alpha T + \beta s$, where T is the temperature and s denotes the salinity. If we take into account one

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of the factors (T or s) or the diffusivities for T and s are equal, then their effect in the dynamics can be represented simply by the density ρ, or equivalently the buoyancy $b = \rho_0 - \rho = \alpha T - \beta s$ (see [24]). This is why we prefer to deal with the variables u and b. In order to simplify our mathematical presentation we impose on u and b the periodic boundary conditions of the same type as in [23] (see also [24] and the references therein). Thus we consider our problem in the domain

$$\mathcal{O} = (0, L_1) \times (0, L_2) \times (-L_3/2, L_3/2) \subset \mathbb{R}^3$$

and denote the spatial variable in $\mathcal{O}$ by $\bar{x} = (x, z) = (x_1, x_2, z) \in \mathcal{O}$. We use the notation ∇, div and Δ for the gradient, divergence and Laplace operators in two dimensional variable $x = (x_1, x_2)$. Below we also denote by $\Delta_{x,z}$ the 3D Laplace operator. The notations $\nabla_{x,z}$ and $\operatorname{div}_{x,z}$ have a similar meaning.

We consider the following equations for the fluid velocity field

$$u = (v(\bar{x}, t); w(\bar{x}, t)) = (v^1(\bar{x}, t); v^2(\bar{x}, t); w(\bar{x}, t)), \quad \bar{x} = (x, z),$$

for the pressure $P = P(\bar{x}, t)$ and for the buoyancy $b = b(\bar{x}, t)$:

$$v_t + (v, \nabla)v + w\partial_z v - \nu \Delta v - \nu \partial_{zz} v + f v^\perp + \nabla P = G_f \quad \text{in} \quad \mathcal{O} \times (0, +\infty), \quad (1)$$

$$\operatorname{div} v + \partial_z w = 0 \quad \text{in} \quad \mathcal{O} \times (0, +\infty), \qquad (2)$$

$$\partial_z P = b \quad \text{in} \quad \mathcal{O} \times (0, +\infty), \qquad (3)$$

$$b_t + (v, \nabla)b + w\partial_z b - \nu \Delta b - \nu \partial_{zz} b = G_b \quad \text{in} \quad \mathcal{O} \times (0, +\infty), \qquad (4)$$

where $\nu > 0$ is the dynamical viscosity, G_f and G_b are volume sources, and f is the Coriolis parameter. We denote $v^\perp = (-v^2; v^1)$. We supplement (1)–(4) with the periodic boundary conditions imposed on $(v; w)$, P and b. We also assume v and P are even with respect to z and w and b are odd with respect to z (hence $w\big|_{z=0} = 0$ and $b\big|_{z=0} = 0$). These requirements and also (2), (3) lead to the following relations

$$w(\bar{x}, t) = -\int_0^z \operatorname{div} v(x, \xi, t)d\xi, \quad P(\bar{x}, t) = p(x, t) + \int_0^z b(x, \xi, t)d\xi \qquad (5)$$

for every $\bar{x} = (x, z) \in \mathcal{O}$. It is important (see [3] for the basic discussion) that the pressure p is independent of the vertical variable z and depends on 2D (horizontal) variable x only. Exploring intensively this observation the authors of the paper [3] have implemented a new effective approach for proving of the global well-posedness for problems like (1)–(4), see the discussion below.

The system in (1)–(4) was studied by many authors for the different types of boundary conditions (see the survey [24] and the references therein). The existence of week solutions [22] and (global) well-posedness of strong solutions were established (see [3] and also [16, 17, 23]). The existence of

a global attractor for the viscous 3D primitive equations was proved in [15] (see also the paper [23] devoted to the periodic case). However, to the best of our knowledge, the question concerning dimension of this attractor and the related question on determining modes (functionals) are still open. We note that the author of [15] claimed the finiteness of the dimension in 2007 with the reference to a forthcoming paper which is not published yet.

It should be also noted that stochastic perturbations of (1)–(4) were studied in [9, 13, 14] (see also the references in these publications) and the paper [10] deals with a random kick forcing of the primitive equations.

Our goal is to show that the system in (1)–(4) possesses an additional regularity and satisfies a squeezing property in the Ladyzhenskaya form (see [20, 21]). This property implies several important facts about dynamics of the system and, in particular, the finiteness of the fractal dimension of the attractor (see Corollary 3.6) and the existence of finite number of determining modes or functionals (Corollary 3.7). The squeezing property also plays an important role in the study of turbulent behaviour in some classes of dynamical systems with random kicks arising in the fluid dynamics (see, e.g., [18, 19]). In this paper we apply the squeezing property established and also the theory developed in [19] to present a result on ergodicity of (1)–(4) forced by random bounded kicks with appropriately chosen frequency (see Corollary 3.9).

It is commonly recognized (see [3, 16, 17, 23, 24]) that the main difficulty arising in the study of the primitive equations in (1)–(4) is related to the hydrodynamical part. The calculations which should be made to obtain appropriate bounds for a buoyancy type variable are either standard or repeat the corresponding argument for the fluid variables in the simplified form. This is why below we state our results for the full problem (1)–(4), but in the proof we concentrate on the fluid equations only. Moreover, for the transparency of our argument we even assume that the buoyancy equation is absent. This corresponds to the case of the zero buoyancy $(b(\bar{x}, t) \equiv 0)$ which is possible when $G_b \equiv 0$ and $b(\bar{x}, 0) = 0$.

The paper is organized as follows.

Section 2 contains some preliminary material. Here we represent the primitive equations in some standard form, introduce Sobolev type spaces and state the well-known result on well-posedness of strong solutions. In Section 3 we state our main results. The key outcome of Theorem 3.1 is the existence of an absorbing ball lying in an H^2 type space. This allows us not only guarantee the existence of a global attractor but also provides an important step in the proof of the squeezing in Theorem 3.5. In this section we also apply our main results to prove the finiteness of the dimension of the global attractor, the existence of finite number of determining functionals, and ergodicity of a random kicks forced model generated by (1)–(4). In Section 4 we demonstrate the main steps of the proofs of Theorems 3.1 and 3.5 in the case when the buoyancy variable b is absent.

2   Preliminaries

In this section we rewrite system (1)–(4) in the canonical form, introduce appropriate Sobolev type spaces and quote the result on the strong well-posedness.

Using the representations in (5) we arrive at the system

$$v_t + (v, \nabla)v - \left[\int_0^z \operatorname{div} v d\xi\right] \partial_z v - \nu[\Delta v + \partial_{zz} v] + fv^\perp$$

$$= -\nabla\left[p(x,t) + \int_0^z b d\xi\right] + G_f \quad \text{in} \quad \mathcal{O} \times (0, +\infty), \quad (6)$$

$$b_t + (v, \nabla)b - \left[\int_0^z \operatorname{div} v d\xi\right] \partial_z b - \nu[\Delta b + \partial_{zz} b] = G_b \quad \text{in} \quad \mathcal{O} \times (0, +\infty), \quad (7)$$

supplied with the conditions:

$$\operatorname{div} \int_{-L_3/2}^{L_3/2} v dz = 0; \quad v \text{ is periodic in } \bar{x} \text{ and even in } z, \quad \int_{\mathcal{O}} v d\bar{x} = 0; \quad (8)$$

and

$$b \text{ is periodic in } \bar{x} \text{ and odd in } z. \quad (9)$$

We also need to add intial data for v and b:

$$v(0) = v_0, \quad b(0) = b_0. \quad (10)$$

We denote by $\dot{H}^s_{per}(\mathcal{O})$ the Sobolev space of order s consisting of periodic functions such that $\int_{\mathcal{O}} f dx = 0$. To describe fluid velocity fields we introduce the following spaces with $s = 0, 1, 2$:

$$V_s = \left\{v = (v^1; v^2) \in [\dot{H}^s_{per}(\mathcal{O})]^2 : \ v^i \text{ is even in } z, \ \operatorname{div} \int_{-L_3/2}^{L_3/2} v dz = 0\right\}.$$

We also denote $H = V_0$ and equip H with L_2 type norm $\|\cdot\|$ and denote by $(\cdot, \cdot)$ the corresponding inner product. The space V_1 and V_2 are endowed with the norms $\|\cdot\|_{V_1} = \|\nabla_{x,z} \cdot\|$ and $\|\cdot\|_{V_2} = \|\Delta_{x,z} \cdot\|$.

To describe the buoyancy we need the spaces

$$E_s = \left\{b \in \dot{H}^s_{per}(\mathcal{O}) : \ b \text{ is odd in } z\right\}, \quad s = 0, 1, 2,$$

equipped with the standard norms. We also denote $W_s = V_s \times E_s$ with the standard (Euclidean) product norms.

As it was already mentioned, starting with [3] the global well-posedness of the equations in (6) and (7) was studied by many authors [16, 17, 23, 24]. In this section we quote the result from [23] (see also [24]) on well-posedness of strong solutions under the periodic boundary conditions.

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Proposition 2.1 ([23, 24]) *Let $G_f \in V_0$ and $G_b \in E_0$. Then for every $U_0 = (v_0; b_0) \in W_1$ problem (6)–(10) has a unique strong solution $(u(t); b(t))$:*

$$U(t; U_0) \equiv (u(t); b(t)) \in C(\mathbb{R}_+; W_1) \cap L_2(0, T; W_2), \quad \forall T > 0.$$

Moreover, this solution continuously depends on time t and initial data U_0 and generates a dynamical system (S_t, W_1) with the phase space W_1 and the evolution operator S_t defined by the relation $S_t U_0 = U(t; U_0)$. This evolution operator S_t possesses the following Lipschitz property

$$\|S_t U - S_t U_*\|_{W_1} \leq C_{T,R} \|U - U_*\|_{W_1}, \quad t \in [0, T],$$

for every $T > 0$ and $U, U_* \in B(R) \equiv \{U : \|U\|_{W_1} \leq R\}$.

Remark 2.2 Formally [23, 24] do not contain the proofs of continuity of $S_t U$ with respect to t and the Lipschitz property in W_1. However almost the same calculations as in [15] alow us to show these properties. We also note that in the case considered unique global solvability can be also established in smoother classes. For instance, if $G_f \in V_{m-1}$, $G_b \in E_{m-1}$ and $U_0 = (v_0; b_0) \in W_m$ for some $m \geq 2$ then (see [23, 24]) the solution $U(t)$ belongs to the class $C(\mathbb{R}_+; W_m) \cap L_2(0, T; W_{m+1})$ for every $T > 0$. This observation is important in our further considerations due to possibility to use smooth approximations of solutions in our calculations.

We conclude this preliminary section with the statement of the following well-known uniform Gronwall lemma.

Lemma 2.3 ([25]) *Let g, h, y be nonnegative locally integrable functions on $[t_0, +\infty)$ such that*

$$\frac{dy}{dt} \leq gy + h,$$

and

$$\int_t^{t+1} g(s)ds \leq a_1, \quad \int_t^{t+1} h(s)ds \leq a_2, \quad \int_t^{t+1} y(s)ds \leq a_3$$

for any $t \geq t_0$, where $a_i > 0$ are constant. Then

$$y(t+1) \leq (a_3 + a_2)\, e^{a_1} \quad \text{for any } t \geq t_0.$$

3   Results

We start with the following assertion which partially follows from Proposition 2.1 and based on the calculations given in [23].

Theorem 3.1 (Smooth Absorbing Ball) *Let $G_f \in V_1$, $G_b \in E_1$ and also $(\partial_z G_f; \partial_z G_b) \in [L_6(\mathcal{O})]^3$. Then the dynamical system (S_t, W_1) generated by problem (6)–(10) is dissipative and compact[1] Moreover, there exists $R_* > 0$ such that the ball*

$$\mathscr{B} = \{U \in W_2 : \|U\|_{W_2} \leq R_*\}$$

is absorbing, i.e., for any bounded set B in W_1 there is t_B such that

$$S_t B \subset \mathscr{B} \quad \text{for all } t \geq t_B.$$

In this ball $\mathscr{B}$ there is a forward invariant absorbing set $\mathscr{D}$.

We note that the dissipativity and compactness of (S_t, W_1) stated in Theorem 3.1 is known from [23] under slightly weaker conditions on G_f and G_b. However the existence of an absorbing compact set which is *bounded in W_2* requires additional hypotheses and additional calculations.

Remark 3.2 In the case when $G_f \equiv 0$ and $G_b \equiv 0$ one can prove (see some argument below in Section 4.1) that

$$\forall \varepsilon > 0, \quad \exists T(R, \varepsilon) : \quad \|S_t U\|_{W_1} \leq \varepsilon \quad \forall t \geq T(R, \varepsilon), \ \|U\|_{W_1} \leq R.$$

This means that in the case of vanishing forces the zero equilibrium state is asymptotically stable.

Using the standard results (see, e.g., one of the monographs [1, 5, 25]) on the existence of global attractors we can derive from Theorem 3.1 the following assertion.

Corollary 3.3 (Global Attractor) *Let the hypotheses of Theorem 3.1 be in force. Then the dynamical system (S_t, W_1) generated by problem (6)–(10) possesses a compact global attractor $\mathfrak{A}$ which is a bounded set in W_2.*

Remark 3.4 The result on the existence of a global attractor for 3D viscous primitive equations was known before, see [15], and also Remark 2.1 in [23] for the periodic case. Our improvement is that we state the boundedness of $\mathfrak{A}$ in H^2 type Sobolev space. One can also show that in the case of sufficiently smooth sources G_f and G_b the attractor $\mathfrak{A}$ possesses additional spatial smoothness. Using the method presented in [23] we can even prove Gevrey regularity of the elements from the attractor provided the sources possess appropriate smoothness properties. However we do not pursue these improvements because our main goal is a squeezing property with application to dimension and ergodicity.

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[1] For the definitions see [1], for instance.

In the space W_1 we consider the bilinear form

$$a(U, U_*) = \int_{\mathcal{O}} \left[\nabla_{x,z} v \cdot \nabla_{x,z} v_* + \nabla_{x,z} b \cdot \nabla_{x,z} b_* \right] dx dz,$$

where $U = (v; b)$ and $U_* = (v_*; b_*)$ are elements from W_1. This form generates a positive self-adjoint operator A with a discrete spectrum, Due to periodicity of the problem the corresponding eigenfunctions and eigenvalues can be easily described. We assume that they are reorganized in such way that

$$A e_k = \lambda_k e_k, \quad 0 < \lambda_1 \leq \lambda_2 \leq \ldots, \quad \lim_{k \to +\infty} \lambda_k = +\infty.$$

Moreover, $\{e_k\}$ is the orthonormal basis in W_0. We denote by P_N the orthoprojecton on the $\mathrm{Span}\{e_1, \ldots, e_N\}$ and $Q_N = I - P_N$.

Now we can state our main result.

Theorem 3.5 (Squeezing) *Let the hypotheses of Theorem 3.1 be in force. Then the dynamical system (S_t, W_1) generated by problem (6)–(10) on the forward invariant absorbing set $\mathscr{D}$ possesses the properties:*

- **Lipschitz property:** *For any $U, U_* \in \mathscr{D}$ we have the relation*

$$\|S_t U - S_t U_*\|_{W_1} \leq C_{\mathscr{D}} e^{\alpha_{\mathscr{D}} t} \|U - U_*\|_{W_1}, \quad \forall \, t \geq 0, \qquad (11)$$

  where $C_{\mathscr{D}}$ and $\alpha_{\mathscr{D}}$ are positive constants.

- **Squeezing property:** *For every $T > 0$ and $0 < q < 1$ there exists $N = N(T, q)$ such that*

$$\|Q_N[S_T U - S_T U_*]\|_{W_1} \leq q \|U - U_*\|_{W_1} \qquad (12)$$

  for any U and U_* from the absorbing forward invariant set $\mathscr{D}$.

The squeezing property stated in Theorem 3.5 allow us to apply Ladyzhenskaya Theorem on the dimension of invariant sets (see [20, 21] and also Theorem 8.1 in [5, Chap.1]). This yields the following assertion.

Corollary 3.6 (Finite Dimension) *Under the hypotheses of Theorem 3.1 the global attractor $\mathfrak{A}$ of the dynamical system (S_t, W_1) generated by problem (6)–(10) has a finite fractal dimension.*

This means that the asymptotic dynamics in this model is finite-dimensional and topologically can be represented by a compact set in $\mathbb{R}^d$ with appropriate $d < \infty$.

The next outcome of the squeezing property is the existence of a finite number of (asymptotically) determining modes. We note that this notion was introduced in [12] for 2D Navier-Stokes system and was studied by many authors (see the discussion in the recent paper [11]). For a general theory of the determining functionals we refer to [4], see also [5] and [7] for a development of this theory based on the notion of the completeness defect.

Corollary 3.7 (Determining Modes) *Let the hypotheses of Theorem 3.1 be in force and N be such that* (12) *holds for some $T > 0$ and $0 < q < 1$. Then the dynamical system (S_t, W_1) possesses N determining modes. This means that the relation*

$$\lim_{t \to +\infty} \|P_N[S_t U - S_t U_*]\|_{W_1} = 0 \quad \text{for some } U, U_* \in W_1 \qquad (13)$$

implies that $\|S_t U - S_t U_*\|_{W_1} \to 0$ as $t \to +\infty$.

Proof. The relation in (12) implies that

$$\|S_T U - S_T U_*\|_{W_1} \le q\|U - U_*\|_{W_1} + \|P_N[S_T U - S_T U_*]\|_{W_1}, \quad 0 < q < 1, \quad (14)$$

for any U and U_* from the absorbing forward invariant set $\mathscr{D}$. Iterating the relation in (14) we obtain that

$$\|S_T^n U - S_T^n U_*\|_{W_1} \le q^n\|U - U_*\|_{W_1} + \sum_{k=0}^{n-1} q^{n-1-k}\|P_N[S_T^k U - S_T^k U_*]\|_{W_1} \quad (15)$$

for every $n = 1, 2, \ldots$, where U and U_* are from $\mathscr{D}$. This inequality and also the Lipschitz property in (11) makes it possible to prove that relation (13) implies that $\|S_t U - S_t U_*\|_{W_1} \to 0$ as $t \to +\infty$. For some details in the abstract situation we refer to the proof of Theorem 1.3 in [5, Chap.5]. $\qquad \square$

Remark 3.8 Using the same idea as in [7] it is also possible to derive from (15) the existence of other finite families of determining functionals.

Now we consider an example of an application of the results above to ergodicity of a random kick forced model generated by problem (6)–(10). We deal with the simplest situation and suppose that $G_f \equiv 0$ and $G_b \equiv 0$. Moreover, we consider a model in some fixed ball from W_1 and assume that the frequency of kicks is smaller than a certain critical value (depending on the radius of the ball and the amplitude of the kicks). We plan to remove all these restrictions and discuss further developments in our joint studies with S.Kuksin and A.Shirikyan.

Let $\{\eta_k\}$ be i.i.d. random variables in W_1. Assume that the support of the distribution $\mathcal{D}(\eta_1)$ in W_1 contains the origin and is bounded, i.e., there exists $R_{kick} > 0$ such that

$$\operatorname{supp} \mathcal{D}(\eta_1) \subset B(R_{kick}) \equiv \{U \in W_1 : \ \|U\|_{W_1} \le R_{kick}\}.$$

Here and below we keep the notation $B(R)$ for the ball with the center at 0 of the radius R in the space W_1.

Let us fix $\widehat{R} > R_{kick}$ and choose $T_c = T(\widehat{R}, R_{kick}, \mathscr{D}) \ge 1$ such that

$$S_t B(\widehat{R}) \subset B(\widehat{R} - R_{kick}) \quad \text{and} \quad S_{t-1} B(\widehat{R}) \subset \mathscr{D} \quad \text{for all } t \ge T_c,$$

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where $\mathscr{D}$ is the forward invariant absorbing set given by Theorem 3.1. This choice of T_c is possible due to Remark 3.2 and Theorem 3.1.

Now we take $T \geq T_c$ and define a Markov chain in W_1 by the formula

$$U_k = F_k(U_{k-1}) \equiv S_T U_{k-1} + \eta_k, \quad k = 1, 2, \dots \qquad (16)$$

This chain is called a random kicks model generated by (6)–(10) (with $G_f \equiv 0$ and $G_b \equiv 0$). The parameter T^{-1} has meaning of the frequency of the kicks. We refer to [19] for motivation an physical importance of different types of kick models in the turbulence theory.

It is obvious that if $U_0 \in B(\widehat{R})$, then we also have that $U_k \in B(\widehat{R})$ for all k. Thus we also have a chain in $B(\widehat{R})$.

It follows from Proposition 2.1 that

$$\|S_{T-1}U - S_{T-1}U_*\|_{W_1} \leq C_{T,\widehat{R}}\|U - U_*\|_{W_1}, \quad \text{for all } U, U_* \in B(\widehat{R}).$$

Let us fix $0 < q < 1$ such that $\eta = qC_{T,\widehat{R}} < 1$. Since $S_{T-1}U, S_{T-1}U_* \in \mathscr{D}$, we can apply Theorem 3.5 and choose N such that

$$\|Q_N[S_T U - S_T U_*]\|_{W_1} \leq q\|S_{T-1}U - S_{T-1}U_*\|_{W_1} \leq \eta\|U - U_*\|_{W_1}$$

for all $U, U_* \in B(\widehat{R})$. Thus we have the squeezing property on the ball $B(\widehat{R})$.

Applying now Theorem 3.2.5[19] and the properties of the mapping S_t established above we arrive to the following assertion.

Corollary 3.9 (Ergodicity) *Assume that $P_N\eta_1$ and $Q_N\eta_1$ are independent and the distribution of $P_N\eta_1$ in the (finite-dimensional) space $P_N W_1$ has a density $p(U)$ with respect to the Lebesgue measure dU on $P_N W_1$ such that*

$$\int_{P_N W_1} |p(U+V) - p(U)|dU \leq C\|V\|_{W_1}, \quad \forall V \in P_N W_1,$$

where C is a constant. Then for the chain (16) there is a unique invariant probability Borel measure μ on $B(\widehat{R})$. Moreover, there is $0 < \gamma < 1$ such that

$$\|F_k^* \lambda - \mu\|_L^* \leq C(\widehat{R}, \lambda) \cdot \gamma^k, \quad k = 1, 2, \dots$$

for any probability Borel measure λ on $B(\widehat{R})$, where $\| \cdot \|_L^*$ denotes the corresponding dual Lipschitz norm (see, e.g., [19]).

To obtain this result we apply [19, Theorem 3.2.5]. Conditions 3.2.1–3.2.3 in [19] follow from the properties of S_t stated above. We also note that [19, Theorem 3.2.5] formally deals with a Markov chain in a *whole* space. However one can see that the arguments from [19] remain true if we restrict the iteration procedure in (16) on an invariant subset (this observation was already used in [6] to prove ergodicity for random kick-forced 3D Navier-Stokes equations in a thin domain). Another possibility to fix this issue is

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to consider the extension $\widehat{S}_T$ of S_T from $B(\widehat{R})$ on the whole space W_1 by the formula

$$\widehat{S}_T U = \begin{cases} S_T U, & \|U\|_{W_1} \le \widehat{R}; \\ S_T \left(\frac{\widehat{R} U}{\|U\|_{W_1}} \right), & \|U\|_{W_1} > \widehat{R}. \end{cases}$$

One can see that $\widehat{S}_T$ maps W_1 into $B(\widehat{R} - R_{kick})$. Thus after the first step in (16) we are in $B(\widehat{R})$ and therefore we have $S_T = \widehat{S}_T$ for $k \ge 2$ in (16). We also note that $\widehat{S}_T$ is globally Lipschitz (the proof of the latter property can be found in [8, p.64], for instance) and the Conditions 3.2.1–3.2.3 [19] follow directly.

4   Proofs

Our arguments are more or less standard and use the methods developed in [3, 15, 23] (see also the survey [24]). In fact they are some refinement of the calculations known now from [3], see also [15, 23, 24]). This why we consider only key steps in the argument. Moreover, we deal with the reduced system which appears in the following way.

If we assume $G_b \equiv 0$ and $b_0 = 0$, we can take $b \equiv 0$ as a solution to (7). Thus we arrive to the problem: to find a (horizontal) fluid velocity field

$$v(x,t) = (v^1(\bar{x},t); v^2(\bar{x},t)), \quad \bar{x} = (x,z),$$

and the pressure $p(x,t)$ satisfying the equation

$$v_t + (v, \nabla)v - \left[\int_0^z \operatorname{div} v d\xi \right] \partial_z v - \nu[\Delta v + \partial_{zz} v] + f v^\perp = -\nabla p + G_f \quad (17)$$

in $\mathcal{O} \times (0, +\infty)$ supplied with conditions (8) and with initial data

$$v(x,z,0) = v_0(x,z). \tag{18}$$

It follows from Proposition 2.1 that for every $G_f \in V_1$ and $v \in V_1$ problem (17),(18),(8) has a unique strong solution

$$v(t) \in C(\mathbb{R}_+; V_1) \cap L_2(0,T; V_2), \quad \forall T > 0.$$

These solutions generate a dynamical system $(\tilde{S}_t, V_1)$ in V_1. Below we prove our main result for this system. Thus we deal with calculations involving the velocity field v, but not the buoyancy. Moreover using approximation procedure and the results from [23, 24] we can even assume that all our calculations deal with smooth solutions.

4.1   Preliminary a priori estimates

In order to prove Theorem 3.1 for the system $(\tilde{S}_t, V_1)$ generated by (17) we need only to check the existence of absorbing ball in the space V_2. For this we need to repeat and partially refine some calculations in [15, 23].

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4.1.1   A priori estimate in H

If we multiply (17) by v in H, then we obtain

$$\frac{d\|v\|^2}{dt} + \nu\big[\|\nabla v\|^2 + \|\partial_z v\|^2\big] \le c\|G_f\|^2,$$

which implies that

$$\|v(t)\|^2 + \nu \int_t^{t+1} \big[\|\nabla v\|^2 + \|\partial_z v\|^2\big]d\tau \le 2e^{-c_0 t}\|v(0)\|^2 + c_1\|G_f\|^2$$

for all $t \ge 0$. Thus there exists $R_0 > 0$ such that for any $R > 0$ there is $t_R \ge 0$ such that

$$\|v(t)\|^2 + \nu \int_t^{t+1} \big[\|\nabla v\|^2 + \|\partial_z v\|^2\big]d\tau \le R_0^2 \quad \text{for all} \ \ t \ge t_R \qquad (19)$$

with $\|v(0)\| \le R$. Moreover, we can choose $R_0^2 = \varepsilon + c_1\|G_f\|^2$ with arbitrary $\varepsilon > 0$, in this case t_R also depends on ε.

4.1.2   Splitting

Let

$$\bar{v} = \frac{1}{L_3}\int_{-L_3/2}^{L_3/2} v dz \equiv \langle v \rangle_z \ \ \text{and} \ \ \tilde{v} = v - \bar{v}.$$

As in [3] one can see that the fields $\bar{v}$ and $\tilde{v}$ satisfy the equations

$$\bar{v}_t + (\bar{v}, \nabla)\bar{v} + \overline{(\tilde{v}, \nabla)\tilde{v}} + \overline{(\nabla, \tilde{v})\tilde{v}} - \nu\Delta\bar{v} + f\bar{v}^\perp = -\nabla p + \overline{G_f} \qquad (20)$$

with $\operatorname{div} \bar{v} = 0$ in $\mathbb{T}^2 \equiv (0, L_1) \times (0, L_2)$, and

$$\tilde{v}_t + (\tilde{v}, \nabla)\tilde{v} - \left[\int_0^z \operatorname{div} \tilde{v}dz\right]\partial_z\tilde{v}$$
$$+ (\tilde{v}, \nabla)\bar{v} + (\bar{v}, \nabla)\tilde{v} - \overline{[(\tilde{v}, \nabla)\tilde{v} + \overline{(\nabla, \tilde{v})\tilde{v}}]}$$
$$- \nu[\Delta\tilde{v} + \partial_{zz}\tilde{v}] + f\tilde{v}^\perp = \widetilde{G_f} \quad (21)$$

in $\mathcal{O} = \mathbb{T}^2 \times (-L_3/2, L_3/2)$.

4.1.3 H^1-estimates

Now we multiply (21) by $|\tilde{v}|^4\tilde{v}$ and integrate over $\mathcal{O}$. In the same way as in [3, 15] we obtain

$$\frac{d}{dt}\|\tilde{v}\|_{L_6}^6 + \nu\int_{\mathcal{O}} \big[|\nabla\tilde{v}|^2 + |\partial_z\tilde{v}|^2\big]|\tilde{v}|^4 d\bar{x}$$
$$\le C_0\big[\|\bar{v}\|^2\|\nabla\bar{v}\|^2 + \|\nabla\tilde{v}\|^2\big]\|\tilde{v}\|_{L_6}^6 + C_1\|G_f\|_{L_6}\|\tilde{v}\|_{L_6}^5.$$

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This implies that

$$\frac{d}{dt}\|\tilde{v}\|_{L_6}^2 \leq C_0\left[1 + \|v\|^2\right]\|\|\nabla v\|^2 \cdot \|\tilde{v}\|_{L_6}^2 + C_1\|G_f\|_{L_6}\|\tilde{v}\|_{L_6}$$

$$\leq C_0\left[1 + (1 + \|v\|^2)\|\|\nabla v\|^2\right] \cdot \|\tilde{v}\|_{L_6}^2 + C_1\|G_f\|_{L_6}^2.$$

Since $\|\tilde{v}\|_{L_6} \leq c\left[\|\nabla v\|^2 + \|\partial_z v\|^2\right]$, we can apply Lemma 2.3 and relation (19) to obtain that there exists $R_1 \geq R_0 > 0$ such that for any $R > 0$ there is $t_R^* \geq t_R$ such that

$$\|\tilde{v}(t)\|_{L_6} \leq R_1 \quad \text{for all} \ \ t \geq t_R^* \quad \text{with} \ \ \|v(0)\| \leq R. \tag{22}$$

Moreover using (19) with $R_0^2 = \varepsilon + c_1\|G_f\|^2$ we can conclude that

$$\forall \varepsilon \in (0,1]: \quad \|\tilde{v}(t)\|_{L_6}^2 \leq C\left[\varepsilon + \|G_f\|^2\right] \quad \text{for all} \ \ t \geq t_{R,\varepsilon}^*$$

with $\|v(0)\| \leq R$. Using (22) we can also assume that

$$\int_t^{t+1} \int_{\mathcal{O}} \left[|\nabla\tilde{v}|^2 + |\partial_z\tilde{v}|^2\right]|\tilde{v}|^4 d\bar{x} d\tau \leq C(R_1) \quad \text{for all} \ \ t \geq t_R^* \tag{23}$$

with $\|v(0)\| \leq R$. In the case $G_f \equiv 0$ we can change $C(R_1)$ into $C\varepsilon$ and t_R^* into $t_{R,\varepsilon}^*$.

Next we multiply (20) by $-\Delta\bar{v}$ in $L_2(\mathbb{T}^2)$. As in [3] using the relations

$$\int_{\mathbb{T}^2} (\bar{v}, \nabla)\bar{v}\Delta\bar{v}dx = 0, \quad \int_{\mathbb{T}^2} \nabla p\Delta\bar{v}dx = 0, \quad \int_{\mathbb{T}^2} \bar{v}^\perp\Delta\bar{v}dx = 0,$$

we have that

$$\frac{1}{2}\frac{d}{dt}\|\nabla\bar{v}\|^2 + \nu\|\Delta\bar{v}\|^2 = \int_{\mathbb{T}^2} \left[\overline{(\tilde{v}, \nabla)\tilde{v}} + \overline{(\nabla, \tilde{v})\tilde{v}}\right]\Delta\bar{v}dx - \int_{\mathbb{T}^2} \overline{G_f}\Delta\bar{v}dx.$$

This implies (see [3] for some details) that

$$\frac{d}{dt}\|\nabla\bar{v}\|^2 + \nu\|\Delta\bar{v}\|^2 \leq C\left[\|\nabla\tilde{v}\|^2 + \int_{\mathcal{O}} |\tilde{v}|^4|\nabla\tilde{v}|^2 d\bar{x} + \|G_f\|^2\right].$$

Therefore (22) and (23) give us that there exists $R_2 > 0$ such that for any $R > 0$ there is $t_R^{**} \geq t_R^*$ such that

$$\|\nabla\bar{v}(t)\|^2 + \nu\int_t^{t+1} \|\Delta\bar{v}\|^2 d\tau \leq R_2^2 \quad \text{for all} \ \ t \geq t_R^{**} \quad \text{with} \ \ \|v(0)\| \leq R. \tag{24}$$

As above in the case $G_f \equiv 0$ we can take R_2^2 of the order ε (with t_R^{**} depending on ε).

The next step is the estimate for v_z. We first note that $u \equiv v_z$ solves the problem

$$u_t + (v, \nabla)u - \left[\int_0^z \operatorname{div} v d\xi\right]\partial_z u - \nu[\Delta u + \partial_{zz} u]$$

$$= - fu^\perp - (u, \nabla)v + (\nabla, v)u + \partial_z G_f \tag{25}$$

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in the class of periodic (*odd* in z) functions. Since $u^\perp u = 0$ and

$$\int_{\mathcal{O}} \left\{ (v, \nabla)u - \left[\int_0^z \operatorname{div} v d\xi \right] \partial_z u \right\} u d\bar{x} = 0,$$

using the multiplier u after integration by parts we obtain that

$$\frac{1}{2} \frac{d\|u\|^2}{dt} + \nu \left[\|\nabla u\|^2 + \|\partial_z u\|^2 \right] \leq \|\partial_z G_f\| \|u\| + \int_{\mathcal{O}} |v| |u| |\nabla u| d\bar{x},$$

which implies (for some details we refer to [3]) that

$$\frac{d\|u\|^2}{dt} + \nu \left[\|\nabla u\|^2 + \|\partial_z u\|^2 \right] \leq C \left[\|v\|_{L_6}^4 \|u\|^2 + \|\partial_z G_f\|^2 \right]$$
$$\leq C \left[\|\tilde{v}\|_{L_6}^4 + \|\nabla \bar{v}\|^4 \right] \|u\|^2 + C \|\partial_z G_f\|^2.$$

Therefore there exists $R_3 > 0$ such that for any $R > 0$ there is $t_R^{(3)} \geq t_R^{**}$ such that

$$\|v_z(t)\|^2 + \nu \int_t^{t+1} \left[\|\nabla v_z\|^2 + \|v_{zz}\|^2 \right] d\tau \leq R_3^2 \quad \text{for all} \quad t \geq t_R^{(3)}$$

with $\|v(0)\| \leq R$. Moreover, $R_2^2 \sim \varepsilon$ in the case $G_f \equiv 0$.

Next we estimate $\|\nabla v\|$. For this we multiply (17) by $-\Delta v$. As in [3] we obtain

$$\frac{d\|\nabla v\|^2}{dt} + \nu \left[\|\Delta v\|^2 + \|\nabla \partial_z v\|^2 \right]$$
$$\leq C \left[\|v\|_{L_6}^4 + \|\nabla v\|^2 \|v_z\|^2 \right] \|\nabla v\|^2 + C \|G_f\|^2$$

which implies again that there exists $R_4 > 0$ such that for any $R > 0$ there is $t_R^{(4)} \geq t_R^{(3)}$ such that

$$\|\nabla v(t)\|^2 + \nu \int_t^{t+1} \left[\|\Delta v\|^2 + \|\nabla v_z\|^2 + \|v_{zz}\|^2 \right] d\tau \leq R_4^2 \qquad (26)$$

for all $t \geq t_R^{(4)}$ with $\|v(0)\| \leq R$. Moreover one can see from the analysis above that in the case $G_f \equiv 0$ for any $0 < \varepsilon < 1$ there exists $t_{R,\varepsilon} > 0$ such that

$$\|\nabla v(t)\|^2 + \nu \int_t^{t+1} \left[\|\Delta v\|^2 + \|\nabla v_z\|^2 + \|v_{zz}\|^2 \right] d\tau \leq \varepsilon \quad \text{for all} \quad t \geq t_{R,\varepsilon} \quad (27)$$

provided $\|v(0)\| \leq R$.

The relation in (26) yields that the system $(\tilde{S}_t, V_1)$ is dissipative in V_1. In the case $G_f \equiv 0$ by (27) an absorbing ball can be chosen with an arbitrary small radius which implies that the zero equilibrium is asymptotically stable in this case (as it is claimed in Remark 3.2 when the buoyancy is presented).

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4.2   Completion of the proof of Theorem 3.1

To conclude the proof we need to show uniform smoothing of trajectories. We note that the result for individual trajectories is known from [23] (see also [24]). However in these references the size of the corresponding ball is not controlled and thus we need some refinement of that results. This is why below we mainly follow the line of argument in [23] (see also [24]).

We multiply (17) by $(-\Delta_{x,z})^2 v$ to get

$$\frac{1}{2}\frac{d}{dt}\|\Delta_{x,z}v\|^2 + \nu\|[-\Delta_{x,z}]^{3/2}v\|^2 = -\int_{\mathcal{O}}[(v,\nabla)v + w(v)\partial_z v]\,(-\Delta_{x,z})^2 v d\bar{x}$$
$$+ (G_f, (-\Delta_{x,z})^2 v), \qquad (28)$$

where $w(v) = w(\bar{x}, t)$ is defined by (5). We also use the facts that

$$\int_{\mathcal{O}} v^{\perp}(-\Delta_{x,z})^2 v d\bar{x} = \int_{\mathcal{O}} [\Delta_{x,z}v]^{\perp}\Delta_{x,z}v d\bar{x} = 0$$

and

$$\int_{\mathcal{O}} \nabla p (-\Delta_{x,z})^2 v d\bar{x} = 0.$$

The general structure of the nonlinear terms in (28) is given by the formula

$$N(v) \equiv \int_{\mathcal{O}} [(v,\nabla)v + w(v)\partial_z v]\,(-\Delta_{x,z})^2 v d\bar{x}$$

$$= \int_{\mathcal{O}} \Delta_{x,z}[(v,\nabla)v + w(v)\partial_z v]\,\Delta_{x,z}v d\bar{x}$$

$$= \int_{\mathcal{O}} [(v,\nabla)\Delta_{x,z}v + w(v)\partial_z\Delta_{x,z}v]\,\Delta_{x,z}v d\bar{x} + \Sigma(v) = \Sigma(v).$$

Here

$$\Sigma(v) \sim \sum_{k,i,j}\int_{\mathcal{O}}\left[D^2 v^i D^1 v^j D^2 v^j + D^k w(v)D^{2-k}\partial_z v^j\Delta_{x,z}v^j\right] d\bar{x}, \qquad (29)$$

where $k, i, j = 1, 2$, D^k is a differential operator of order k, and the sign $\sim$ means that we deal with linear combination of the terms in the right hand side.

Using the embeddings $H^1(\mathcal{O}) \subset L_6(\mathcal{O})$ and $H^{1/2}(\mathcal{O}) \subset L_3(\mathcal{O})$ (see [26]) and also interpolation one can see that

$$\Sigma_1 \equiv \left|\int_{\mathcal{O}} D^2 v^i D^1 v^j \Delta_{x,z}v^j d\bar{x}\right| \leq C\|D^2 v_i\|\|D^1 v_j\|_{L_6}\|D^2 v_j\|_{L_3}$$

$$\leq C\|v\|_{H^2}^2\|D^2 v^j\|^{1/2}\|D^2 v^j\|_{H^1}^{1/2} \leq \varepsilon\|[-\Delta_{x,z}]^{3/2}v\|^2 + C_\varepsilon\|v\|_{H^2}^{10/3}$$

To estimate the second term in (29) we start with the case $k = 2$:

$$\int_{\mathcal{O}} D^2 w(v)\partial_z v^j\Delta_{x,z}v^j d\bar{x} \leq C\|D^2 w(v)\|\|\partial_z v^j\|_{L_6}\|D^2 v^j\|_{L_3}$$

$$\leq C\|v\|_{H^3}^{3/2}\|\partial_z v\|_{L_6}\|v\|_{H^2}^{1/2}$$

$$\leq \varepsilon\|[-\Delta_{x,z}]^{3/2}v\|^2 + C_\varepsilon\|\partial_z v\|_{L_6}^4\|v\|_{H^2}^2.$$

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To estimate the term with $k = 1$ in (29) we use the following lemma (see Proposition 2.2 in [2]).

Lemma 4.1 *For smooth functions ϕ and ψ and vector field $v = (v^1; v^2)$ we have*

$$\left| \int_{\mathcal{O}} \left[\int_0^z \operatorname{div} v(x, \xi) d\xi \right] \psi \phi d\bar{x} \right| \leq C \|v\|_{H^2}^{1/2} \|v\|_{H^1}^{1/2} \|\psi\|_{H^1}^{1/2} \|\psi\|^{1/2} \|\phi\|.$$

This lemma yields the following estimate

$$\int_{\mathcal{O}} D^1 w(v) D^1 \partial_z v^j \Delta_{x,z} v^j d\bar{x} \leq C \|v\|_{H^2}^{1/2} \|v\|_{H^3}^{1/2} \|\partial_z v^j\|_{H^2}^{1/2} \|\partial_z v^j\|_{H^1}^{1/2} \|D^2 v\|$$

$$\leq C \|v\|_{H^3} \|v\|_{H^2}^2 \leq \varepsilon \|[-\Delta_{x,z}]^{3/2} v\|^2 + C_\varepsilon \|v\|_{H^2}^4.$$

Thus we arrive at the following relation

$$\frac{d}{dt} \|\Delta_{x,z} v\|^2 + \nu \|[-\Delta_{x,z}]^{3/2} v\|^2 \leq C_1 \left[1 + \|\Delta_{x,z} v\|^2 + \|\partial_z v\|_{L_6}^4 \right] \|\Delta_{x,z} v\|^2$$
$$+ C_2 \|G_f\|_{V_1}^2. \tag{30}$$

Our next step is an estimate for $\|\partial_z v\|_{L_6}$. As above let $u = \partial_z v$. We multiply (25) by $|u|^4 u$. Using the relations $u^\perp u = 0$ and

$$\int_{\mathcal{O}} \left\{ (v, \nabla) u - \left[\int_0^z \operatorname{div} v d\xi \right] \partial_z u \right\} |u|^4 u d\bar{x} = 0,$$

we obtain that

$$\frac{1}{6} \frac{d}{dt} \|u\|_{L_6}^6 + \nu \int_{\mathcal{O}} \left[|\nabla u|^2 + |\partial_z u|^2 \right] |u|^4 d\bar{x}$$

$$\leq \int_{\mathcal{O}} \{-(u, \nabla) v + (\nabla, v) u\} |u|^4 u d\bar{x} + (\partial_z G_f, |u|^4 u).$$

In the same way as in [23, 24] after integration by parts we have that

$$\int_{\mathcal{O}} \{-(u, \nabla) v + (\nabla, v) u\} |u|^4 u d\bar{x} \leq C \|v\|_{L_\infty} \int_{\mathcal{O}} |\nabla u| |u|^5 d\bar{x}$$

$$\leq C \|v\|_{L_\infty} \left[\int_{\mathcal{O}} |\nabla u|^2 |u|^4 d\bar{x} \right]^{1/2} \|u\|_{L_6}^3$$

$$\leq \varepsilon \int_{\mathcal{O}} |\nabla u|^2 |u|^4 d\bar{x} + C_\varepsilon \|v\|_{L_\infty}^2 \|u\|_{L_6}^6.$$

This implies that

$$\frac{d}{dt} \|u\|_{L_6}^6 + \nu \int_{\mathcal{O}} \left[|\nabla u|^2 + |\partial_z u|^2 \right] |u|^4 d\bar{x}$$

$$\leq C \left[\|v\|_{H^2}^2 \|u\|_{L_6}^6 + \|\partial_z G_f\|_{L_6} \|u\|_{L_6}^5 \right].$$

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Thus we have that

$$\frac{d}{dt}\|u\|_{L_6}^2 \le C_1 \left[1 + \|\Delta_{x,z}v\|^2\right]\|u\|_{L_6}^2 + C_2\|\partial_z G_f\|_{L_6}^2.$$

Using Lemma 2.3 and the estimate in (26) we obtain that

$$\|\partial_z v(t)\|_{L_6} \le R_5 \quad \text{for all} \quad t \ge t_R^{(5)} \ge t_R^{(4)} \quad \text{with} \quad \|v(0)\| \le R.$$

Now returning to (30) and applying Lemma 2.3 allow us obtain the following assertion.

Proposition 4.2 *Let $G_f \in V_1$ and $\partial_z G_f \in L_6(\mathcal{O})$. Then there exists $R_* > 0$ such that the ball $\mathscr{B} = \{v \in V_2 : \|v\|_{V_2} \le R_*\}$ is absorbing for the dynamical system $(\tilde{S}_t, V_1)$ generated by problem (17). Moreover, there is a forward invariant absorbing set $\mathscr{D}$ in this ball.*

Thus the proof of Theorem 3.1 (in the case $b_0 \equiv 0$ and $G_b \equiv 0$) is complete.

4.3 Squeezing estimate

Now we prove Theorem 3.5 (in the case $b_0 \equiv 0$ and $G_b \equiv 0$). We start with the following assertion.

Proposition 4.3 *For any v_1, v_2 from the absorbing forward invariant set $\mathscr{D}$ we have the relation*

$$\|\tilde{S}_t v_1 - \tilde{S}_t v_2\|_{V_1} \le C_{\mathscr{D}} e^{\alpha_{\mathscr{D}} t}\|v_1 - v_2\|_{V_1}, \quad \forall t \ge 0,$$

where $C_{\mathscr{D}}$ and $\alpha_{\mathscr{D}}$ are positive constants. Moreover, for $v_*(t) = \tilde{S}_t v_*$ and $v_{**}(t) = \tilde{S}_t v_{**}$ we also have that

$$\|u(t)\|_{V_1}^2 + \int_0^t \|u(\tau)\|_{V_2}^2 d\tau \le C_{\mathscr{D}} e^{\alpha_{\mathscr{D}} t}\|v_* - v_{**}\|_{V_1}^2, \quad \forall t \ge 0,$$

where $u(t) = v_*(t) - v_{**}(t)$.

Proof. We use the same type of argument as in [15]. For this we write the equation

$$u_t - \nu[\Delta u + \partial_{zz} u] + (u, \nabla)v_* + (v_*, \nabla)u$$
$$- \left[\int_0^z \operatorname{div} u d\xi\right]\partial_z v_* - \left[\int_0^z \operatorname{div} v_{**} d\xi\right]\partial_z u + f u^\perp = -\nabla(p_* - p_{**}) \quad (31)$$

for the difference $u(t) = v_*(t) - v_{**}(t) = S_t v_* - S_t v_{**}$. Then we multiply (31) by $(-\Delta_{x,z})u$ and apply the same argument as in [15]. At the final stage we use the fact that

$$\|v_*(t)\|_{V_2} + \|v_{**}(t)\|_{V_2} \le C_{\mathscr{D}}, \quad t > 0,$$

for initial data v_* and v_{**} from $\mathscr{D}$. □

Let $\{\lambda_k\}$ be the eigenvalues of the corresponding Stokes type operator A which arises in problem (17) and $\{e_k\}$ be the corresponding eigenvectors. We can assume that

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots; \quad \lim_{k\to\infty} \lambda_k = \infty.$$

We denote by P_N the orthoprojector in H on $\mathrm{Span}\{e_1, \ldots, e_N\}$. Let $Q_N = I - P_N$. We multiply (31) by $Q_N A u$ and obtain that

$$\frac{1}{2}\frac{d}{dt}\|A^{1/2}Q_N u\|^2 + \nu\|AQ_N u\|^2 = -([(u,\nabla)v_* + (v_{**},\nabla)u], Q_N A u)$$
$$+ \int_{\mathcal{O}}\left[\int_0^z \mathrm{div}\, u\, d\xi\right]\partial_z v_* Q_N A u\, d\bar{x} - f(u^\perp, Q_N A u)$$
$$+ \int_{\mathcal{O}}\left[\int_0^z \mathrm{div}\, v_{**}\, d\xi\right]\partial_z u Q_N A u\, d\bar{x}. \tag{32}$$

Now we need the following lemma which follows from the Hölder inequality and from the standard Sobolev embedding (see, e.g., [26]):

$$H^s(\mathcal{O}) \subset L_p(\mathcal{O}) \quad \text{when} \quad s - 3/2 = -3/p, \quad p \geq 2, \quad \mathcal{O} \subset \mathbb{R}^3.$$

Lemma 4.4 *Let $\mathcal{O} \subset \mathbb{R}^3$ and u, v be sufficiently smooth functions. Then*

$$\|uv\|_{L_2} \leq C\|u\|_{H^s}\|v\|_{H^r}$$

for any $0 < s, r < 3/2$. such that $r + s = 3/2$.

Using this lemma to estimate the right hand side in (32) on the forward invariant set $\mathscr{D}$ we can find that

$$\frac{1}{2}\frac{d}{dt}\|A^{1/2}Q_N u\|^2 + \nu\|AQ_N u\|^2 \leq C_{R_*}\|u\|_{H^{3/2}}\|AQ_N u\|,$$

where R_* is the same as in the statement of Proposition 4.2. This yields

$$\frac{d}{dt}\|A^{1/2}Q_N u\|^2 + \nu\|AQ_N u\|^2 \leq C_{R_*,\varepsilon}\|u\|_{V_1}^2 + \varepsilon\|u\|_{V_2}^2$$

for any $\varepsilon > 0$. This implies that

$$\|A^{1/2}Q_N u(t)\|^2 \leq e^{-\nu\lambda_{N+1}t}\|A^{1/2}Q_N u(0)\|^2$$
$$+ C_{R_*,\varepsilon}\int_0^t e^{-\nu\lambda_{N+1}(t-\tau)}\|u\|_{V_1}^2\, d\tau + \varepsilon\int_0^t e^{-\nu\lambda_{N+1}(t-\tau)}\|u\|_{V_2}^2\, d\tau.$$

Using Proposition 4.3 we obtain that

$$\|A^{1/2}Q_N u(t)\|^2 \leq \left[e^{-\nu\lambda_{N+1}t} + \left(\frac{C_{R_*,\varepsilon}}{\alpha_{R_*} + \nu\lambda_{N+1}} + \varepsilon C_{R_*}\right)e^{\alpha_{R_*}t}\right]\|A^{1/2}u(0)\|^2.$$

Now for any fixed $T > 0$ and $0 < q < 1$ we can choose ε and N such that

$$\|A^{1/2}Q_N u(T)\|^2 \leq q^2 \|A^{1/2}u(0)\|^2.$$

Thus we arrive to the squeezing property: For every $T > 0$ and $0 < q < 1$ there is N such that

$$\|Q_N[\tilde{S}_T v_1 - \tilde{S}_T v_2]\|_{V_1} \leq q\|v_1 - v_2\|_{V_1}$$

for any v_1, v_2 from the absorbing forward invariant set $\mathscr{D}$.

Acknowledgement

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